Reconstructing trajectories from the moments of occupation measures

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Some common difficulties:

- Many local optima;
- Handling of state constraints;
- Existence of solution(s) not guaranteed;
- Need of expert supervision.

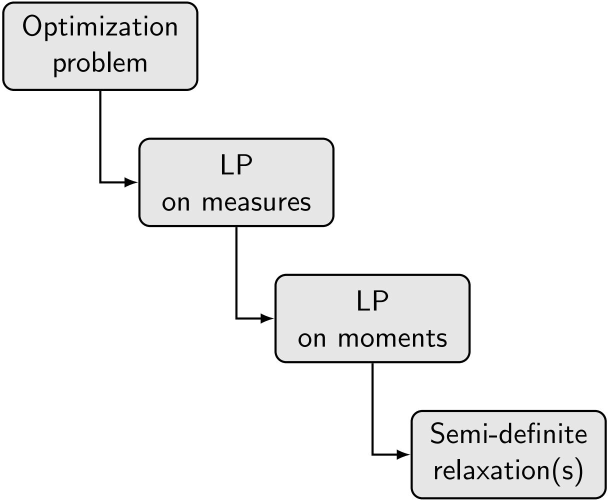
Convex approach to optimal control

- 1940s-1950s: control (Young, Filippov, . . .).
- 1970s-1980s: whole problem (Vinter, Lewis, Rubio, . . .).

Nb: approach dual to dynamic programming.

Moment approach to optimization

[Lasserre, SICON'01]:



This talk

- Equip [Lasserre et al: SICON'08] with an automated reconstruction technique;
- Method inspired by approach in [Rubio: '86];
- Results in a fully automated, user friendly, numeric approach.

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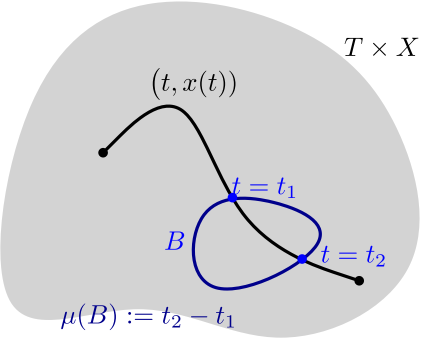
Occupation measures

Objective: capture solutions of $\dot{x} = f(x)$ by *linear* objects.



$$\forall v \in C(T \times X): \quad \langle v(t, \underline{x}), \mu \rangle := \int_0^T v\big(t, x(t)\big)\, \mathrm{d}t$$

Convex relaxations

The optimal control problem:

$$\begin{aligned} J = \inf_{u(t)} & \int_{t_i}^{t_f} h(t,x,u)\,\mathrm{d}t \\ \text{s.t. } & \dot{x} = f(t,x,u), \\ & x(t_i),\ x(t_f) \text{ given}, \\ & x(t) \in \mathbf{X}, \quad u(t) \in \mathbf{U}, \quad t \in \mathbf{T} := [t_i, t_f], \end{aligned}$$

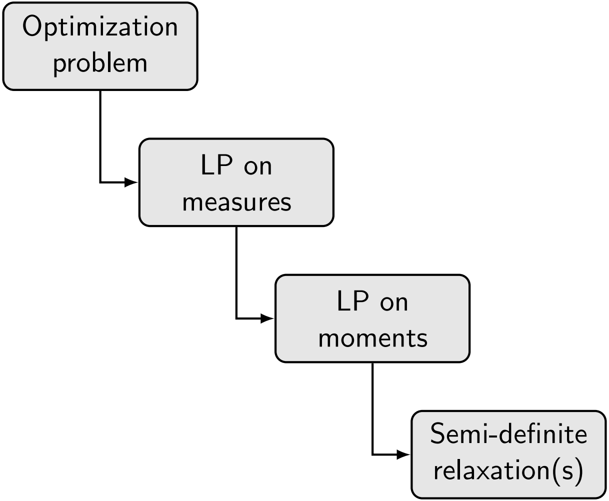
Its convex relaxation:

$$\begin{aligned} J_{LP} = \inf_{\mu} & \ \langle h, \mu \rangle \\ \text{s.t. } & \forall v \in \mathbb{R}[t, \underline{x}] :\ [v(\cdot, x(\cdot))]_{t_i}^{t_f} = \langle \frac{\partial v}{\partial t} + \frac{\partial v}{\partial \underline{x}} f, \mu \rangle, \\ & \mu \in \mathcal{M}^+(\mathbf{T} \times \mathbf{U} \times \mathbf{X}). \end{aligned}$$

Theorem [Vinter: SICON'93]: $J = J_{LP}$ under mild conditions.

Global optimal control

[Lasserre, Henrion, Prieur, Trélat: SICON'08]: use



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The inverse problem

Each relaxation returns (approximate) *moments*

$$y_\alpha := \langle z^\alpha, \mu(\mathrm{d}z) \rangle$$

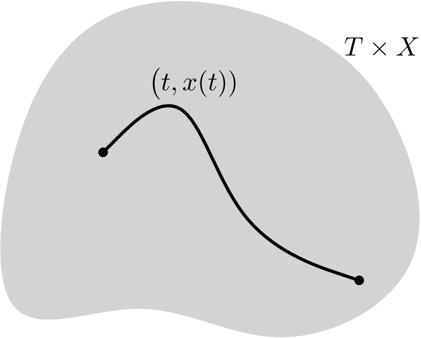
and a (dual) sum-of-square Hamilton-Jacobi-Bellman subsolution.

Problem

Given the moments and SOS certificates, can we reconstruct $(u^*(t), x^*(t))$?

Key idea

Assume solution is unique.
Observation: support of optimal measure = optimal trajectory:



Key idea: reconstruct approximate support of optimal measure.

Tchakaloff's theorem

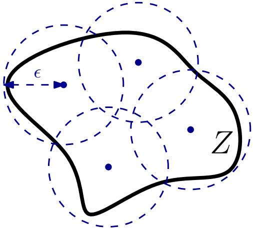
Theorem (Tchakaloff)

Let $\mu \in \mathcal{M}^+(\mathbf{Z} \subset \mathbb{R}^n)$*, and let* $d \geq 1$ *be a fixed positive integer. Then there exists* $p \leq \binom{n+d}{n}$ *points* $z_k \subset \mathbf{Z}$ *and positive weights* λ_k *such that*

$$\langle f, \mu \rangle = \sum_{k=1}^{p} \lambda_k \, f(x_k) \tag{1}$$

for every polynomial $f \in \mathbb{R}[x]$ *of degree at most* d*.*

For compact domain Z, fix discrete mesh Z_ϵ.



Theorem

Consider the following discrete LP:

$$\lambda_\varepsilon^* = \min_{\tilde{\mu},\lambda} \lambda$$
$$\text{s.t. } |y_\alpha - \langle z^\alpha, \tilde{\mu} \rangle| \leq \lambda, \quad \forall \alpha \in \mathbb{N}_{2r}^{1+m+n} \tag{2}$$
$$\tilde{\mu} \in \mathcal{M}^+(\mathbf{Z}_\varepsilon),$$

Then:

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon^* \to 0. \tag{3}$$

Practicalities

Procedure:

1. Solve moment problem for given relaxation order.
2. For each pair (t, x_i) and (t, u_i), solve discrete LP.
3. Assume that non-zero atoms are approximate optimal solution.
4. Use approximate candidate solution as starting guess of local method.
5. If costs agree, certified global solution found. Otherwise, repeat from (1) with higher order.

NB: key difference with [Rubio, 86] is that discrete LPs are used for reconstruction only, and are are low dimensional.

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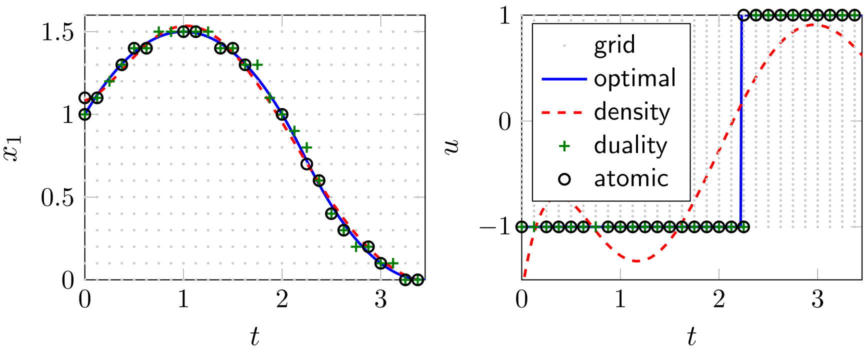
Example 1: double integrator (1/2)

Problem: simple double integrator to origin.

Comparison with other approach:

- Dual approach: for each time t_i in grid, find minimum of SOS certificates to find approximate $(x(t_i)^*, u(t_i)^*)$.
- Use subset of moments to extract polynomial densities as in [Lasserre, Henrion, Mevissen: 13]

Example 1: double integrator (2/2)



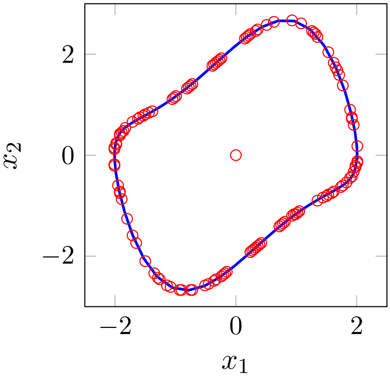
Example 2: invariant measure

Invariant measure:

$$\exists \mu? \text{ s.t. } \forall v \in \mathbb{R}[\underline{x}] : \ \langle \frac{\partial v}{\partial \underline{x}} f, \mu \rangle = 0,$$
$$\langle 1, \mu \rangle = 1,$$
$$\mu \in \mathcal{M}^{+}(\mathbf{X}),$$

Van der Pol oscillator:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + (1 - x_1^2) x_2.$$



Example 3: quadrotor

We consider a 3 DOF quadrotor problem with 7 states, 1 control, and 3 switching modes.

With moment relaxations:

d	p_d^*
2	0.0090
3	0.0943
...	
BOCOP	0.0957

In previous literature: locally optimal values of 0.165 and 0.128.

See [Claeys, Daafouz, Henrion: submitted].

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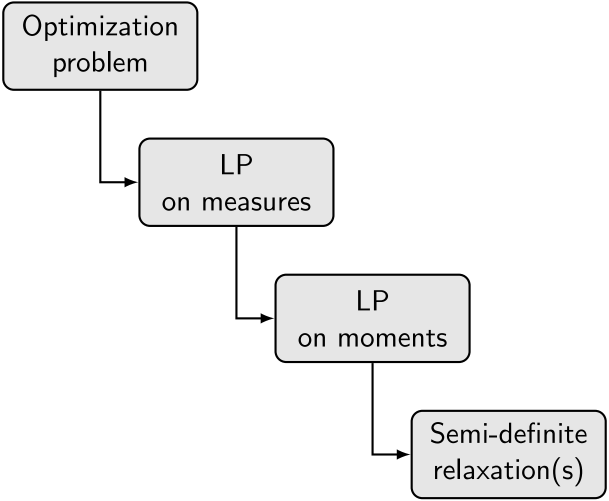
The moment approach



Highlights

- Based solely on convex optimization.
- Code can be fully automated.
- Allows to solve difficult non-linear OCP, also with state-constraints.
- With SDP, can tackle problems up to dimension 6 (NB: on specific problems, one may go higher).
- NB: with approximate SOCP/LP method, [Majumdar et al, CDC'14] have attacked stability problems with 30 states . . .

Thanks!

Presentation available at
http://mathclaeys.wordpress.com