Polynomial optimization and control
Mini-course 3/4: Applications to optimal control

Didier Henrion, Mihai Putinar, Milan Korda, Mathieu Claeys

July 11, 2014

Recap

Optimization problem

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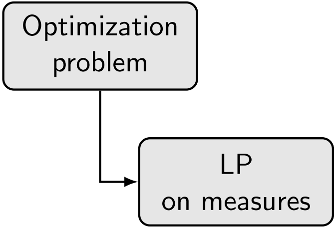

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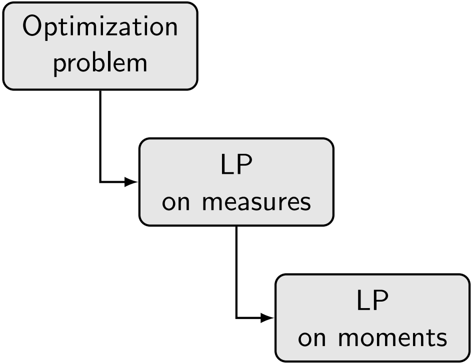

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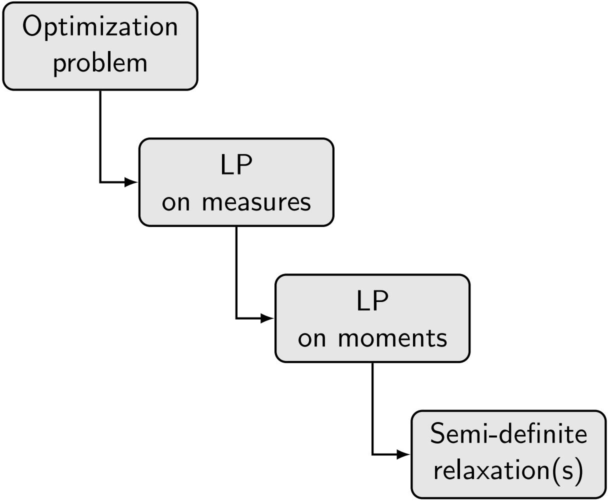
Optimization problem
LP on measures
LP on moments
Semi-definite relaxation(s)

Yesterday's key points...

- *Global* resolution.

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- *Global* resolution.
- Constraints easily captured.
- Moments: a *rich* mathematical history.
- Automated tools (`GloptiPoly`, ...).
- *Many* different applications ...

... today's key points.

- . . . including control !
- MC: open-loop optimal control.
- Milan Korda: closed-loop.

This talk

- How to capture dynamics as linear constraints:
  - bounded control
  - switched systems
  - impulsive systems

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- Applications:
  - Medical imaging
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This talk

- How to capture dynamics as linear constraints:
  - bounded control
  - switched systems
  - impulsive systems
- Applications:
  - Medical imaging
  - Automotive
- Inverse problem.

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The uncontrolled case

$$\inf_{x,T} \int_0^T h(t, x(t))\,\mathrm{d}t$$

$$\text{s.t. } \dot{x} = f(t, x(t))$$

$$x(0) \in X_0$$

$$x(T) \in X_T$$

$$x(t) \in X$$

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\begin{aligned}
&\inf_{x,T} \int_0^T h(t, x(t))\,\mathrm{d}t && \longrightarrow \langle h, \mu \rangle \\
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&\quad x(0) \in X_0 && \\
&\quad x(T) \in X_T && \longrightarrow \mu_T \in \mathcal{M}^+(X_T) \\
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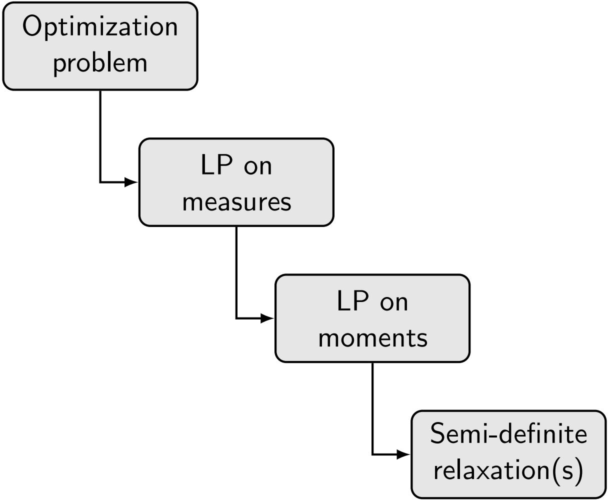
\end{aligned}
$$

Question

How to capture $\{x(t)$ *admissible for ODE* $\}$ *?*

The moment approach



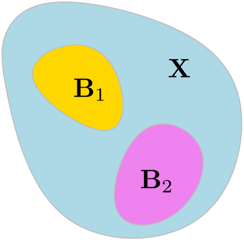
Measures of $\mathbb{R}^n$

- Geometric perspective:

Definition (Finite Borel measures)

$\mu \in \mathcal{M}(\mathbf{X})$ if $\mu : \mathscr{B}(\mathbf{X}) \mapsto \mathbb{R}$ satisfies

- $\mu(\emptyset) = 0$
- $\mu(\mathbf{B}_1 \cup \mathbf{B}_2 \cup \ldots) = \mu(\mathbf{B}_1) + \mu(\mathbf{B}_2) + \ldots$



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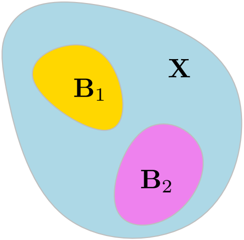

- Functional analysis perspective:

Theorem (Riesz)

$[C(\mathbf{X})]^*$ *"is"* $\mathcal{M}(\mathbf{X})$ *for compact* $\mathbf{X}$.

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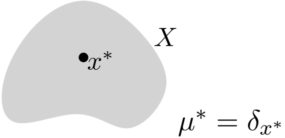

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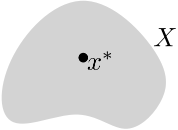
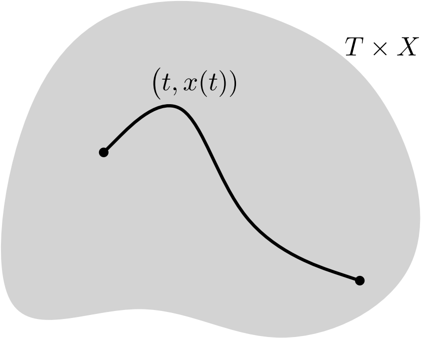


$$\mu^* = \delta_{x^*}$$

- (NB: lift $\neq$ linearization)

Occupation measures

- Geometric:



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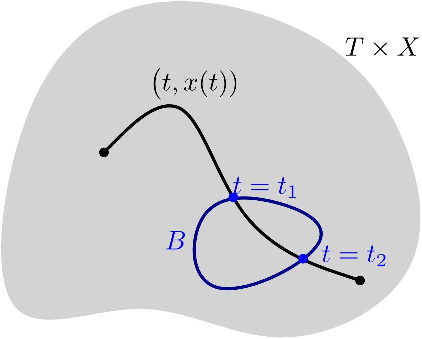

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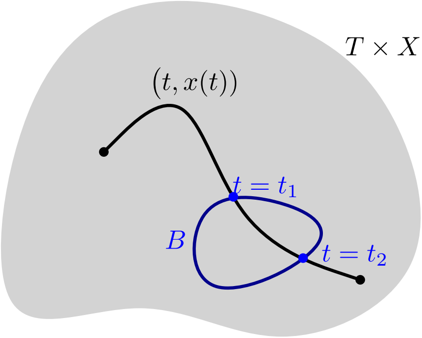

- Functional analysis:

$$\langle v(t,\underline{x}), \mu \rangle = \int_0^T v\big(t, x(t)\big)\, \mathrm{d}t$$

Weak ODE integration

$$v(T, x_T) - v(0, x_0) = \int_0^T \mathrm{d}v(t, x(t))$$

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\begin{aligned}
v(T, x_T) - v(0, x_0) &= \int_0^T \mathrm{d}v(t, x(t)) \\
&= \int_0^T \underbrace{\frac{\partial v}{\partial t}(t, x(t)) + \frac{\partial v}{\partial x}(t, x(t))\, f(t, x(t))}_{:= F(t)} \mathrm{d}t
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Strong and weak sets

Define:

$$\mathscr{S} := \{(\mu, \mu_0, \mu_T) \text{ are occupation measures}\}$$

and

$$\mathscr{W} := \left\{ \begin{array}{c} (\mu, \mu_0, \mu_T) : \\ \langle v, \mu_T \rangle - \langle v, \mu_0 \rangle = \langle \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f, \mu \rangle, \quad \forall v \in \mathcal{C}([0, T] \times X), \\ \langle 1, \mu_0 \rangle = 1 \end{array} \right\}$$

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Theorem (Vinter, Lewis: SICON'78)

$$\mathrm{co}\, \mathscr{S} = \mathscr{W}$$

Global optimal control

[Lasserre, Henrion, Prieur, Trélat: SICON'08]: use

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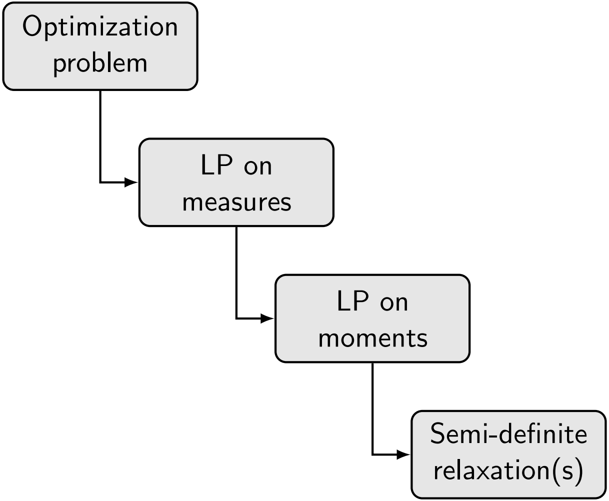

Simplest example (1/3)

$$\inf_{x(t)} \int_0^1 x^2 \, \mathrm{d}t$$

s.t. $\dot{x} = -x$

$$x(0) \in [4, 5]$$
$$x(1) \in [2, 3]$$
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$$x(0) \in [4,5] \quad \longrightarrow \quad \mu_0 \in \mathcal{M}^+([4,5])$$

$$x(1) \in [2,3] \qquad \mu_T \in \mathcal{M}^+([2,3])$$

$$x(t) \in [2,5] \qquad \mu \in \mathcal{M}^+([0,1] \times [2,5])$$

Simplest example (2/3)

Define $\quad y^{\mu}_{\alpha\beta} := \langle t^{\alpha}\, \underline{x}^{\beta}, \mu \rangle, \quad y^{\mu_0}_{\beta} := \langle \underline{x}^{\beta}, \mu_0 \rangle, \quad y^{\mu_T}_{\beta} := \langle \underline{x}^{\beta}, \mu_T \rangle.$

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$$\inf_{(y^\mu, y^{\mu_0}, y^{\mu_T})} y^{\mu}_{02}$$

s.t.

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s.t. $y^{\mu_T}_0 - y^{\mu_0}_0 = 0$ $\quad [v=1]$

$y^{\mu_T}_0 = y^{\mu}_{10}$ $\quad [v=t]$

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$\ldots$

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$\ldots$

$\langle 1, \mu_0\rangle = 1$ $\qquad$ $y^{\mu_0}_0 = 1$

$\mu_0 \in \mathcal{M}^+([4,5])$ $\qquad$ $M(g^{\mu^0}_i * y^{\mu^0}) \succeq 0$

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- With $X_T = [1, 3]$:

$$
\begin{aligned}
J_1^* &= 6.4000 \\
J_2^* &= 6.9173 \\
&\dots \\
J^* &= 6.9173
\end{aligned}
$$

The dual view

Define $\mathcal{L}^* : v \mapsto \mathcal{L}^* v := \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f$.

$$
\begin{aligned}
\inf_{\mu,\mu_0,\mu_T} \; & \langle h, \mu \rangle \\
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$$

dual to

$$
\begin{aligned}
\sup_{r \in \mathbb{R}, v \in C^1} \; & r \\
\text{s.t. } & h + \mathcal{L}^* v \geq 0 \text{ on } K \\
& v - r \geq 0 \text{ on } K_0, \\
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Certificates of given order: dual to moment relaxation of given order.

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Overall strategy:

1. Relax control (Young, Fillipov,. . .)
2. Lift as measure LP (Vinter, Rubio, . . .)
3. Solve by moment relaxations (Lasserre, . . .)

Bounded control (1/2)

Consider $\dot{x} = f(t, x, u)$, $u(t) \in U \subset \mathbb{R}^m$.

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$$\{\omega(\mathrm{d}\underline{u}|t) \in \mathcal{P}(U)\}, \quad [0, T]\text{-a.e}$$

such that $\forall v \in \mathcal{C}(U)$, $t \to \langle v, \omega \rangle$ is measurable on $[0, T]$.

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such that $\forall v \in \mathcal{C}(U)$, $t \to \langle v, \omega \rangle$ is measurable on $[0, T]$.

Example 1: For continuous $u(t)$, pick $\omega = \delta_{u(t)}$, so that $\langle f(t, x(t), \underline{u}), \omega \rangle = f(t, x(t), u(t))$

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Consider $\dot{x} = f(t, x, u)$, $u(t) \in U \subset \mathbb{R}^m$.

Definition (Young measure)

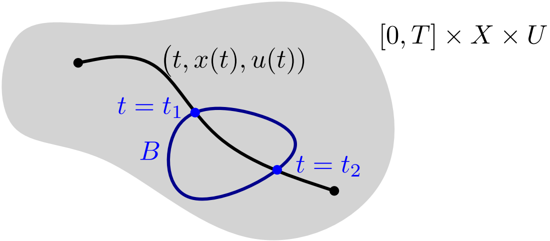
$$\{\omega(\mathrm{d}\underline{u}|t) \in \mathcal{P}(U)\}, \quad [0, T]\text{-a.e}$$

such that $\forall v \in \mathcal{C}(U)$, $t \to \langle v, \omega \rangle$ is measurable on $[0, T]$.

Example 1: For continuous $u(t)$, pick $\omega = \delta_{u(t)}$, so that $\langle f(t, x(t), \underline{u}), \omega \rangle = f(t, x(t), u(t))$

Example 2: Consider a fast, evenly oscillating sequence in $U = \{-1, 1\}$. Tends weakly to $\omega = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$. For $f = u$, $\dot{x} = \langle \underline{u}, \omega \rangle = 0$ exactly.

Occupation measures with control:



Bounded control (2/2)

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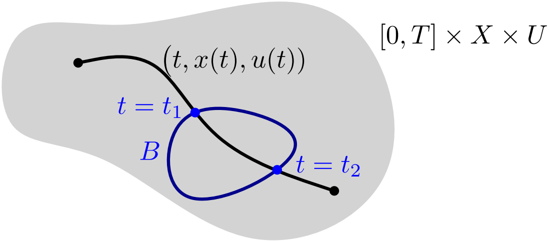

$\mu \in \mathcal{M}^+([0,T]\times X\times U)$ satisfy, $\forall v \in \mathcal{C}([0,T]\times X)$:

$$[v(\cdot,x(\cdot))]_0^T = \langle \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f, \mu \rangle$$

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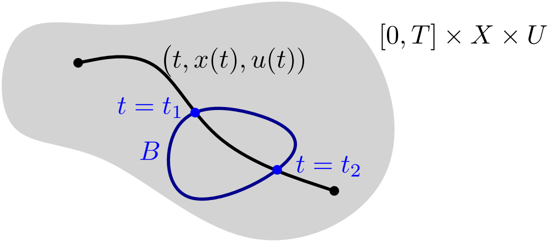

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[Vinter and Lewis, SICON '78]: No relaxation gap if *relaxed* control are considered.

Switched systems (1/2)

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$$\dot{x} = f_{\sigma(t)}(t, x(t)), \qquad \sigma(t) \in \{1, \ldots, m\}$$

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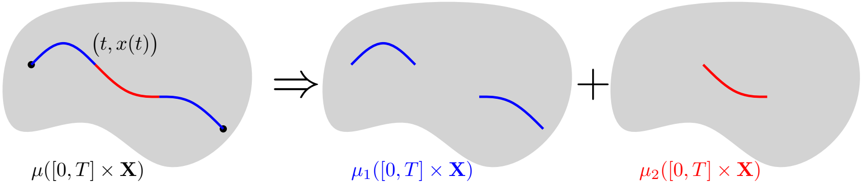
Recast as

$$\dot{x} = \sum_{j=1}^{m} f_j\,(t, x(t))\; u_j(t)$$

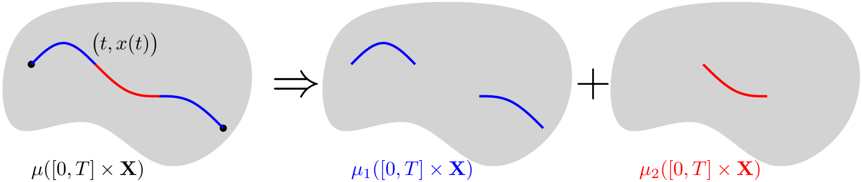
$$u(t) \in \left\{ \underline{u} \in \{0,1\}^m : \sum_{j=1}^{m} \underline{u}_j = 1 \right\}.$$

Modal occupation measures:



Switched systems (2/2)

Modal occupation measures:



Proposition (MC, Daafouz, Henrion: '14)

$$[v(\cdot, x(\cdot))]_0^T = \langle \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \sum_{j=1}^{m} f_j \, u_j, \ \mu(dt, d\underline{x}, d\underline{u}) \rangle$$

$$\Leftrightarrow$$

$$[v(\cdot, x(\cdot))]_0^T = \sum_{j=1}^{m} \langle \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f_j, \ \mu_j(dt, d\underline{x}) \rangle$$

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Consider, with unbounded $u(t)$:

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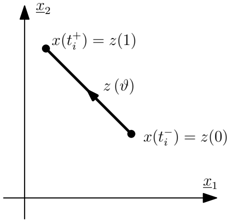

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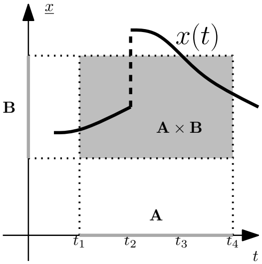

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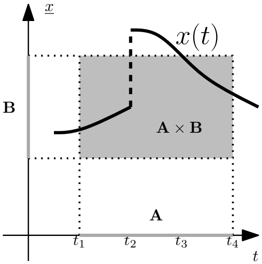

Satisfy:

$$[v(\cdot, x(\cdot))]_0^T = \langle \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} f, \mu \rangle + \langle \frac{\partial v}{\partial x} G, \nu \rangle$$

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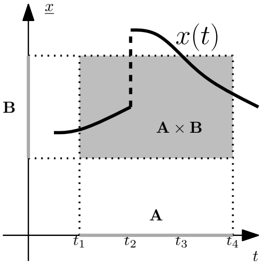

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[MC: thesis '13]
[MC, Arzelier, Henrion, Lasserre: CDC'13] LTV case

Other systems . . .

Stochastic systems:

- [Fleming and Vermes, SICON '89], [Bhatt and Borkar, Ann. Prob. '96], [Kurtz, Stockbridge: SICON '98] for convex lift.
- [Lasserre, "Moments, positive polynomials..."] for some applications in finance via moment relaxations.
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Concentration and oscillations (material science applications):

- DiPerna-Majda measures as control relaxations.
- [MC, Kruzik and Henrion, MTNS '14] solve by moment programming.

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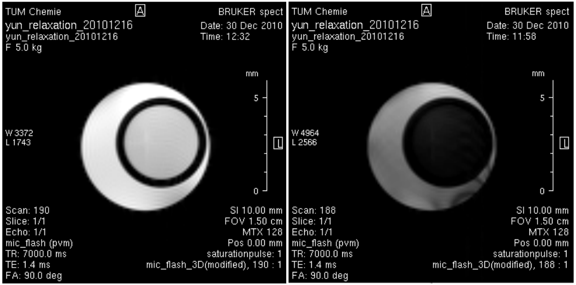
Example: contrast problem (1/4)



Example: contrast problem (2/4)

- [Bonnard, MC, Cots, Martinon: Acta Math. App. '14]

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$$\inf \; -x_3^2(T) - x_4^2(T)$$
$$\text{s.t. } \dot{x}_1 = -\Gamma_1 x_1 - x_2\, u$$
$$\dot{x}_2 = \gamma_1(1 - x_2) + x_1\, u$$
$$\dot{x}_3 = -\Gamma_2 x_3 - x_4\, u$$
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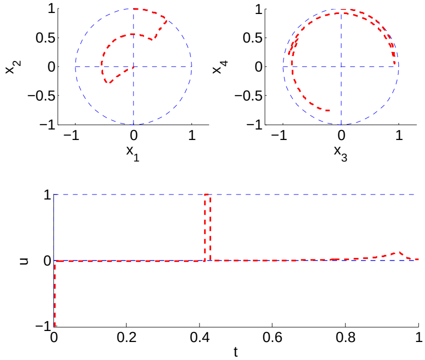
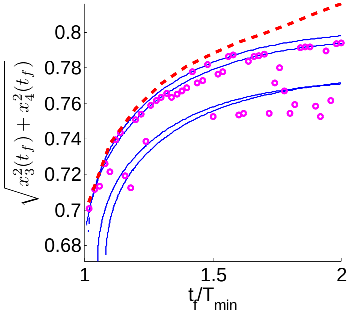


Example: contrast problem (3/4)



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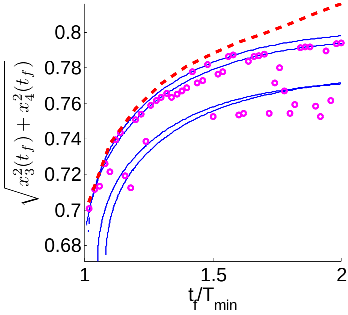

r	Measured control $\sqrt{-J_M^r}$	Measured control t_r	Control measure $\sqrt{-J_M^r}$	Control measure t_r
1	1.000	1	0.9827	0.6
2	0.8984	2	0.8756	1.0
3	0.8707	9	0.8599	6.6
4	0.8256	265	0.7973	113
5	0.7881	5147	0.7891	1298
6	0.7867	50027	0.7871	10831

Example: contrast problem (4/4)

Complexity as $r \to \infty$ of [Lasserre et al. '08]: $\mathcal{O}(r^{\frac{9}{2}(1+n+m)})$

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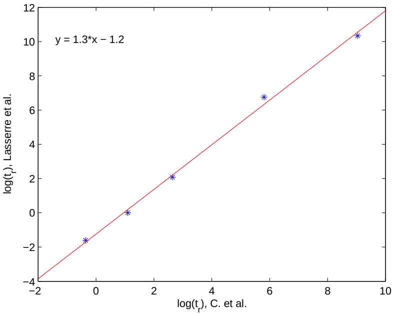

Example: electric car (1/2)

- [Sager, MC, Messine: JOGO'14]

$$\inf_{u(t)} \int_0^{10} (V_{alim}\, x_0 u + R_{bat}\, x_0^2)\, dt$$

$$\text{s.t. } \dot{x}_0 = -\frac{R_m}{L_m} x_0 - \frac{K_m}{L_m} x_1 + \frac{V_{alim}}{L_m} u,$$

$$\dot{x}_1 = \frac{K_m}{J} x_0 - \frac{r M g K_f}{J K_r} - \frac{r^3 \rho S C_x}{2 J K_r^3} x_1^2,$$

$$\dot{x}_2 = \frac{r}{K_r} x_1,$$

$$|x_0(t)| \leq I_{\max}$$

$$u(t) \in \{-1, \ +1\},$$

$$x_2(10) - x_2(0) = 100.$$

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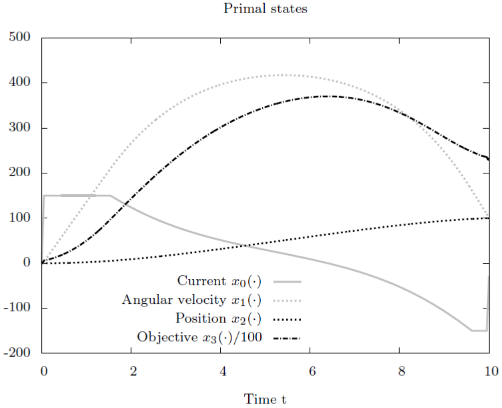

Example: electric car (2/2)

r	Measured control	Control measure
1	0.5	0.5
2	1.0	1.2
3	4.7	3.0
4	12	3.5
5	63	7.8
6	997	23

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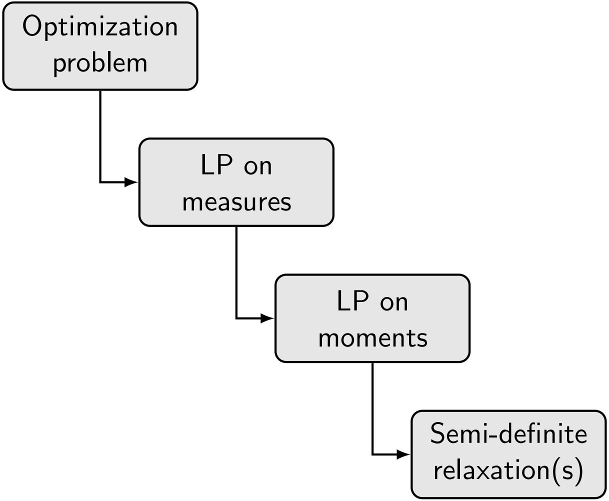
The moment approach



Inverse problem

Given $\{y_\alpha\}_{|\alpha|\leq 2r}$ and dual SOS variables, can we reconstruct $(u^*(t), x^*(t))$?

Method 1: duality

Dual object $V \in \mathbb{R}_{2r}[t, \underline{x}]$ is HJB subsolution:

$$h - \frac{\partial V}{\partial t} - \frac{\partial V}{\partial \underline{x}} f \geq 0 \tag{1}$$

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Procedure:

1. Fix time grid, fix state-control grid
2. For each t_i, find (x_j^*, u_j^*) minimizing LHS of (1).

Method 2: polynomial density

[Henrion, Lasserre, Mevissen. App. Math. Optim. '13].

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Assume $y_{k0\ldots010\ldots0} = \langle t^k \, z(t), \lambda \rangle$

Then polynomial $\tilde{z}(t)$ approaching $z(t)$ in the mean squared sense is found by solving a simple linear system.

Method 3: atomic approximations

[MC, CDC '14]

Support of occupation measure = optimal trajectory(ies)/control(s).

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3. Find best atomic approximation $\tilde{\mu} \in \mathcal{M}^+(\mathbf{Z}_\varepsilon)$ on the grid via

$$\begin{aligned} \lambda^*_\varepsilon = \min_{\tilde{\mu},\lambda} \; & \lambda \\ \text{s.t. } & |y_\alpha - \langle z^\alpha, \tilde{\mu} \rangle| \leq \lambda \end{aligned}$$

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$$\lambda^*_\varepsilon = \min_{\tilde{\mu},\lambda} \lambda$$
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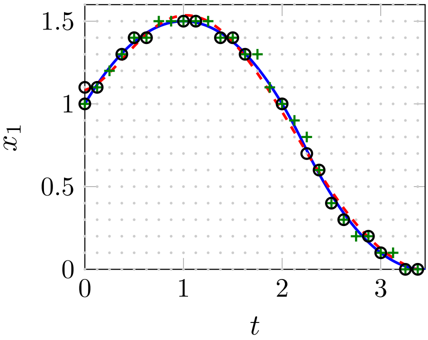
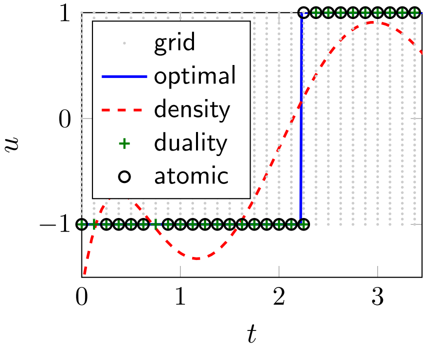
4. Approximate support = non-zero atoms.

Example 1



1.5
1
0.5
0
x_1
0
1
2
3
t



1
0
−1
u
grid
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0
1
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Example 2: invariant measure

Invariant measure:

$$\exists \mu? \text{ s.t. } \forall v \in \mathbb{R}[\underline{x}] : \; \langle \frac{\partial v}{\partial \underline{x}} f, \mu \rangle = 0,$$
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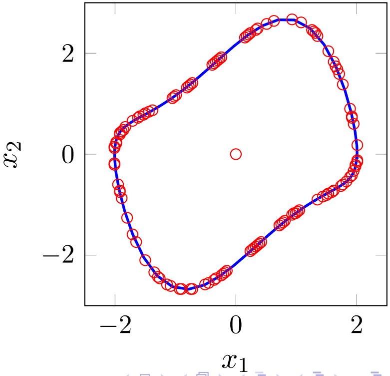

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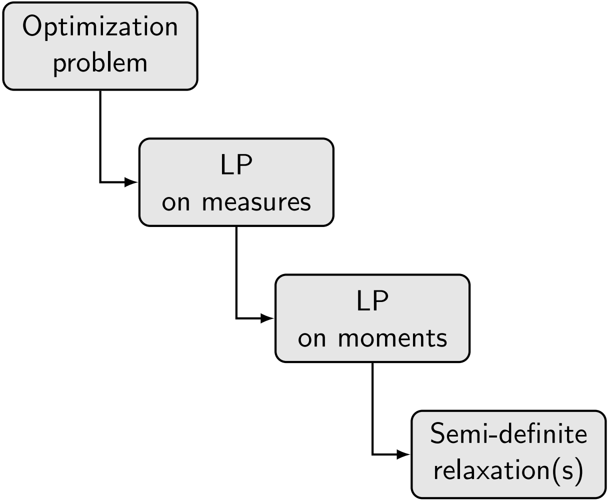

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Thanks!

Presentation available at
`http://mathclaeys.wordpress.com`